ODE

Date 14.7.2020

$$\therefore \cancel{\left(\frac{1}{t}z'\right)'' + \frac{2(n+1)}{t^2}z'' + \frac{z'}{t^2} - \frac{2(n+1)}{t^2}\frac{z'}{t} + \frac{z'}{t^2} = 0}$$

$$1.\ \frac{z'''}{t} - \frac{2}{t^2}z'' + \frac{2}{t^2}z' + \frac{2(n+1)}{t^2}z'' - \frac{4(n+1)}{t^2}\frac{z'}{t}$$

$$+ \frac{2}{t^2}z'' - \frac{2}{t^2}z' \qquad + \frac{2(n+1)}{t^2}\frac{z'}{t} - \frac{2(n+1)}{t^2}\frac{z'}{t} + \frac{z'}{t} = 0$$

$$\left(\frac{1}{t}z'\right)'' + \frac{2(n+2)}{t}\left(\frac{1}{t}z'\right)' + \frac{z''}{t} = 0$$

$$\therefore z_{n+1} = \frac{-1}{t}\frac{d}{dt}z_n \quad \leftarrow \text{can be negative}$$

$$\therefore z_n = (-1)^n\left(\frac{1}{t}\frac{d}{dt}\right)^n z_0$$

$$\frac{x_n}{t^n} = (-1)^n\left(\frac{1}{t}\frac{d}{dt}\right)^n \frac{\sin x}{x} \quad \left(\text{or } \frac{\cos x}{x}\right)$$

$$x_n = (-t)^n\left(\frac{1}{t}\frac{d}{dt}\right)^n \frac{\sin x}{x}$$

---

note that $t^2x'' + 2tx' + t^2x = 0$

let $x = t^2z$, $t^2(t^2z'' + 4tz' + 2z) + 2t(t^2z +$

No.

Date 14 . 7 . 2020

ODE

Associated Legendre polynomial

$$(1-t^2)x'' - 2tx' + \left[l(l+1) - \frac{m^2}{1-t^2}\right]x = 0$$

if $m=0$, it goes back to Legendre equation

$$(1-t^2)x'' - 2tx' + l(l+1)x = 0$$

$$x = P_l(t)$$

$$(1-t^2)^2 x'' - 2t(1-t^2)x' + l(l+1)(1-t^2)x - m^2 x = 0$$

$$x = \sum C_k t^k \quad , \quad x' = \sum (k+1) C_{k+1} t^k \quad , \quad x'' = \sum (k+1)(k+2) C_{k+2} t^k$$

$$(1-t^2)^2 x'' = (1-2t^2+t^4) \sum (k+1)(k+2) C_{k+2} t^k$$

$$= \sum (k+1)(k+2) C_{k+2} (t^k - 2t^{k+2} + t^{k+4})$$

$$= \sum \big[(k+1)(k+2) C_{k+2} \qquad k \text{ from } 0$$

$$- 2k(k-1) C_k \qquad k \text{ from } 2$$

$$+ (k-3)(k-2) C_{k-2} \big] t^k \qquad k \text{ from } 4$$

$$2t(1-t^2)x' = (2t-2t^3) \sum (k+1) C_{k+1} t^k$$

$$= \sum (k+1) C_{k+1} (2t^{k+1} - 2t^{k+3})$$

$$= \sum \big[2k C_k - 2(k-2) C_{k-2} \big] t^k$$

k from 1 $\qquad$ k from 3

ODE

Date 14 · 7 · 2020

$$(1-t^2)\ell(\ell+1)x = (1-t^2)\ell(\ell+1)\sum c_k t^k$$

$$= \sum \ell(\ell+1)\left[\underbrace{c_k}_{k \text{ from } 0} - \underbrace{c_{k-2}}_{k \text{ from } 2}\right]t^k$$

$$-m^2 x = -m^2 \sum c_k t^k$$

$$\therefore (k+1)(k+2)c_{k+2}$$

$$= \left[2k(k-1) + 2k - \ell(\ell+1)\right]c_k + \left[-2(k-2) - (k-2)(k-3) + \ell(\ell+1)\right]c_{k-2}$$

$$+ m^2 c_k \qquad , k \geq 4$$

~~$k=0$, $2c_2 = -\ell(\ell+1)c_0$~~

~~$\therefore c_2 = -\frac{\ell(\ell+1)}{1\cdot 2}c_0$, same as Legendre~~ poly.

~~$k=1$, $2\cdot 3 c_3 = 2c_1$~~

$k=0$, $1\cdot 2 c_2 + \ell(\ell+1)c_0 - m^2 c_0 = 0$

$$c_2 = \frac{m^2 - \ell(\ell+1)}{1\cdot 2}c_0$$

$k=1$, $2\cdot 3 c_3 - 2c_1 + \ell(\ell+1)c_1 - m^2 c_1 = 0$

$$c_3 = \frac{m^2 - \ell(\ell+1) + 2}{2\cdot 3}c_1$$

$k=2$, $3\cdot 4 c_4 - 2\cdot 2\cdot 1 c_2 - 2\cdot 2 c_2 + \ell(\ell+1)c_2 - \ell(\ell+1)c_0$

$$-m^2 c_2 = 0$$

No.

Date 14 . 7 . 2020

ODE

$$3\cdot4C_4 - 8C_2 + l(l+1)C_2 - m^2C_2 - l(l+1)C_0 = 0$$

$$C_4 = \frac{1}{3\cdot4}\left[\left(8+m^2-l(l+1)\right)\frac{m^2-l(l+1)}{1\cdot2}C_0 + l(l+1)C_0\right]$$

a better approach

$m = 0$,

$$(1-t^2)x'' - 2tx' + l(l+1)x = 0$$

$$-2tx'' + (1-t^2)x''' - 2x' - 2tx'' + l(l+1)x' = 0$$

note that

$$(\sqrt{1-t^2}\,x')' = -\frac{t}{\sqrt{1-t^2}}x' + \sqrt{1-t^2}\,x''$$

$$(\sqrt{1-t^2}\,x')'' = -\frac{1}{\sqrt{1-t^2}}x' - \frac{t^2}{\sqrt{1-t^2}^{\,3}}x' - \frac{2t}{\sqrt{1-t^2}}x'' + \sqrt{1-t^2}\,x'''$$

$$= -\frac{1}{\sqrt{1-t^2}^{\,3}}x' - \frac{2t}{\sqrt{1-t^2}}x'' + \sqrt{1-t^2}\,x'''$$

multiply by $\sqrt{1-t^2}$,

$$\sqrt{1-t^2}^{\,3}x''' - 4t\sqrt{1-t^2}\,x'' - 2\sqrt{1-t^2}\,x' + l(l+1)x'\sqrt{1-t^2} = 0$$

also,

$$(1-t^2)(\sqrt{1-t^2}\,x')'' = \sqrt{1-t^2}^{\,3}x''' - 2t\sqrt{1-t^2}\,x'' - \frac{1}{\sqrt{1-t^2}}x'$$

$$-2t(\sqrt{1-t^2}\,x')' = -2t\sqrt{1-t^2}\,x'' + \frac{2t^2}{\sqrt{1-t^2}}x'$$

let $P_l'(t) = \sqrt{1-t^2}\,x'$

$$(1-t^2)P_l''' - 2tP_l'' + \left(\frac{1}{\sqrt{1-t^2}} - \frac{2t^2}{\sqrt{1-t^2}} - 2\sqrt{1-t^2}\right)x' + l(l+1)P_l' = 0$$

$$\therefore\ (1-t^2)P_l''' - 2tP_l'' + \left[l(l+1) - \frac{1}{1-t^2}\right]P_l' = 0$$

No.

Date 14. 7. 2020 ODE

$P_l^1(t) = \sqrt{1-t^2}\,x'$ is a solution for $m=1$.

in general,

$$(1-t^2)x'' - 2tx' + \left[l(l+1) - \frac{m^2}{1-t^2}\right]x = 0$$

$$-2tx'' + (1-t^2)x''' - 2x' - 2tx'' + l(l+1)x' - \frac{m^2}{1-t^2}x' - \frac{2m^2t}{(1-t^2)^2}x = 0$$

$$(1-t^2)^{3/2}x''' - 4t\sqrt{1-t^2}\,x'' + [-2 + l(l+1)]\sqrt{1-t^2}\,x'$$
$$- \frac{m^2}{\sqrt{1-t^2}}x' - \frac{2m^2t}{\sqrt{1-t^2}^3}x = 0$$

remember that

$$(1-t^2)(\sqrt{1-t^2}\,x')'' = \sqrt{1-t^2}^3x''' - 2t\sqrt{1-t^2}\,x'' - \frac{1}{\sqrt{1-t^2}}x'$$

$$-2t(\sqrt{1-t^2}\,x')' = -2t\sqrt{1-t^2}\,x'' + \frac{2t^2}{\sqrt{1-t^2}}x'$$

$$\therefore (1-t^2)(\sqrt{1-t^2}\,x')'' - 2t(\sqrt{1-t^2}\,x')' + l(l+1)\sqrt{1-t^2}\,x'$$
$$\underbrace{\left(\frac{1}{\sqrt{1-t^2}} - \frac{2t^2}{\sqrt{1-t^2}} - 2\sqrt{1-t^2} - \frac{m^2}{\sqrt{1-t^2}}\right)}_{A}x' - \underbrace{\frac{2m^2t}{\sqrt{1-t^2}^3}}x = 0$$

$$A = \frac{-1-m^2}{\sqrt{1-t^2}}, \quad A - \frac{2m^2t}{\sqrt{1-t^2}^3} = -\frac{2m^2t + (1+m^2)(1-t^2)}{\sqrt{1-t^2}^3}$$

this is a huge problem, how to eliminate it?

~~$l(l+1)$~~

ODE

No.

Date 14 . 7 . 2020

note that $\left(\frac{1}{\sqrt{1-t^2}}x'\right)'' = \left(\frac{t}{\sqrt{1-t^2}^3}x' + \frac{1}{\sqrt{1-t^2}}x''\right)'$

Experiment

$$= \left(\frac{1}{\sqrt{1-t^2}^3}x' + \frac{3t^2}{\sqrt{1-t^2}^5}x' + \frac{t}{\sqrt{1-t^2}^3}x'' + \frac{t}{\sqrt{1-t^2}^3}x'' + \frac{1}{\sqrt{1-t^2}}x'''\right)$$

$$= \frac{1+2t^2}{\sqrt{1-t^2}^5}x' + \frac{2t}{\sqrt{1-t^2}^3}x'' + \frac{1}{\sqrt{1-t^2}}x'''$$

$$\left(\frac{1}{\sqrt{1-t^2}}x'\right)' = \frac{t}{\sqrt{1-t^2}^3}x' + \frac{1}{\sqrt{1-t^2}}x''$$

$$\because \sqrt{1-t^2}\,x''' - \frac{4t}{\sqrt{1-t^2}}x'' + [-2+\ell(\ell+1)]\frac{x'}{\sqrt{1-t^2}} - \frac{m^2}{\sqrt{1-t^2}^3}x' - \frac{2m^2t}{\sqrt{1-t^2}^5}x = 0$$

$\therefore$ ~~$(1-t^2)\left(\frac{1}{\sqrt{1-t^2}}x\right)''$~~

$$\left(\frac{t}{\sqrt{1-t^2}}\right)'' = \frac{3t}{\sqrt{1-t^2}^3} + \frac{3t^3}{\sqrt{1-t^2}^5} = \frac{3}{\sqrt{1-t^2}^5}$$

$$\left(\frac{t}{\sqrt{1-t^2}}\right)' = \frac{1}{\sqrt{1-t^2}} + \frac{t^2}{\sqrt{1-t^2}^3} = \frac{1}{\sqrt{1-t^2}^3}$$

$$\left(\frac{t}{\sqrt{1-t^2}}x'\right)'' = \left(\frac{3t}{\sqrt{1-t^2}^3} + \frac{3t^3}{\sqrt{1-t^2}^5}\right)x' + \left(\frac{2}{\sqrt{1-t^2}} + \frac{2t^2}{\sqrt{1-t^2}^3}\right)x'' + \frac{t}{\sqrt{1-t^2}}x'''$$

$$\left(\frac{t}{\sqrt{1-t^2}}x'\right)' = \left(\frac{1}{\sqrt{1-t^2}} + \frac{t^2}{\sqrt{1-t^2}^3}\right)x' + \frac{t}{\sqrt{1-t^2}}x''$$

for $m=2$,

note that $((1-t^2)x'')' = -2tx'' + (1-t^2)x'''$

$$((1-t^2)x'')'' = -2x'' - 2tx''' - 2tx''' + (1-t^2)x''''$$

$$= -2x'' - 4tx''' + (1-t^2)x''''$$

$\cancel{= (1-t^2)((1-t^2)x'')'' - 2t(}$

and, $-2x'' - 2tx''' + (1-t^2)x'''' - 2tx''' - 2x'' - 2x'' - 2tx''' + l(l+1)x'' = 0$

$$(1-t^2)x'''' - 6tx''' - 6x'' + l(l+1)x'' = 0$$

$$(1-t^2)^2x'''' - 6t(1-t^2)x''' - 6(1-t^2)x'' + l(l+1)(1-t^2)x'' = 0$$

$$\therefore (1-t^2)((1-t^2)x'')'' - 2t((1-t^2)x'')' + 4t^2x'' - 4(1-t^2)x'' + l(l+1)(1-t^2)x'' = 0$$

$$(1-t^2)P_l^{2''} - 2tP_l^{2'} + \left[l(l+1) - \frac{4}{1-t^2}\right]P_l^2 = 0$$

by induction,

$$(1-t^2)P_l^{m''} - 2tP_l^{m'} + \left[l(l+1) - \frac{m^2}{1-t^2}\right]P_l^m = 0$$

$$\boxed{P_l^m = (1-t^2)^{\frac{m}{2}} \frac{d^m}{dt^m} P_l}$$

$((1-t^2)^{\frac{m}{2}}x'')' = -mt(1-t^2)^{\frac{m}{2}-1}x'' + (1-t^2)^{m/2}x'''$, $x = P_l^{(m-2)}$

$((1-t^2)^{\frac{m}{2}}x'')'' = -m(1-t^2)^{\frac{m}{2}-1}x'' + m(m-2)t^2(1-t^2)^{\frac{m}{2}-2}x''$

$$- 2mt(1-t^2)^{m/2-1}x''' + (1-t^2)^{\frac{m}{2}}x''''$$

ODE Date 14 · 7 · 2020

and, $\because (1-t^2)x'' - 2tx' + l(l+1)x = 0$

$$(1-t^2)x^{(m+2)} - m \cdot 2t\, x^{(m+1)} - \frac{m(m-1)}{2} \cdot 2x^{(m)}$$

$$-2t x^{(m+1)} - 2m x^{(m)} + l(l+1)x^{(m)} = 0$$

$$(1-t^2)x^{(m+2)} - 2t(m+1)x^{(m+1)} - m(m+1)x^{(m)} + l(l+1)x^{(m)} = 0$$

multiply by $(1-t^2)^{m/2}$

$$(1-t^2)^{m/2+1}x^{(m+2)} - 2t(m+1)(1-t^2)^{m/2}x^{(m+1)}$$

$$- m(m+1)(1-t^2)^{m/2}x^{(m)} + l(l+1)(1-t^2)^{\frac{m}{2}}x^{(m)} = 0$$

$$\therefore (1-t^2)P_l^{m\,''} + m(1-t^2)^{\frac{m}{2}}x^{(m)} - m(m-2)t^2(1-t^2)^{\frac{m}{2}-1}x^{(m)}$$

$$-2tP_l^{m\,'} - 2mt^2(1-t^2)^{\frac{m}{2}-1}x^{(m)} - m(m+1)P_l^m + l(l+1)P_l^m = 0$$

$$(1-t^2)P_l^{m\,''} - 2tP_l^{m\,'} + l(l+1)P_l^m - m^2\frac{1}{1-t^2}P_l^m = 0 \quad //$$